



5th MIDDLE EUROPEAN MATHEMATICAL OLYMPIAD  
VARAŽDIN 2011 CROATIA

language: English

## 5<sup>th</sup> Middle European Mathematical Olympiad

TEAM COMPETITION

4<sup>th</sup> SEPTEMBER 2011

### Problem T-1.

Find all functions  $f: \mathbb{R} \rightarrow \mathbb{R}$  such that the equality

$$y^2 f(x) + x^2 f(y) + xy = xyf(x+y) + x^2 + y^2$$

holds for all  $x, y \in \mathbb{R}$ , where  $\mathbb{R}$  is the set of real numbers.

### Problem T-2.

Let  $a, b, c$  be positive real numbers such that

$$\frac{a}{1+a} + \frac{b}{1+b} + \frac{c}{1+c} = 2.$$

Prove that

$$\frac{\sqrt{a} + \sqrt{b} + \sqrt{c}}{2} \geq \frac{1}{\sqrt{a}} + \frac{1}{\sqrt{b}} + \frac{1}{\sqrt{c}}.$$

### Problem T-3.

For an integer  $n \geq 3$ , let  $\mathcal{M}$  be the set  $\{(x, y) \mid x, y \in \mathbb{Z}, 1 \leq x \leq n, 1 \leq y \leq n\}$  of points in the plane. ( $\mathbb{Z}$  is the set of integers.)

What is the maximum possible number of points in a subset  $S \subseteq \mathcal{M}$  which does not contain three distinct points being the vertices of a right triangle?

**Problem T-4.**

Let  $n \geq 3$  be an integer. At a MEMO-like competition, there are  $3n$  participants, there are  $n$  languages spoken, and each participant speaks exactly three different languages.

Prove that at least  $\left\lceil \frac{2n}{9} \right\rceil$  of the spoken languages can be chosen in such a way that no participant speaks more than two of the chosen languages.

( $\lceil x \rceil$  is the smallest integer which is greater than or equal to  $x$ .)

**Problem T-5.**

Let  $ABCDE$  be a convex pentagon with all five sides equal in length. The diagonals  $AD$  and  $EC$  meet in  $S$  with  $\angle ASE = 60^\circ$ . Prove that  $ABCDE$  has a pair of parallel sides.

**Problem T-6.**

Let  $ABC$  be an acute triangle. Denote by  $B_0$  and  $C_0$  the feet of the altitudes from vertices  $B$  and  $C$ , respectively. Let  $X$  be a point inside the triangle  $ABC$  such that the line  $BX$  is tangent to the circumcircle of the triangle  $AXC_0$  and the line  $CX$  is tangent to the circumcircle of the triangle  $AXB_0$ . Show that the line  $AX$  is perpendicular to  $BC$ .

**Problem T-7.**

Let  $A$  and  $B$  be disjoint nonempty sets with  $A \cup B = \{1, 2, 3, \dots, 10\}$ . Show that there exist elements  $a \in A$  and  $b \in B$  such that the number  $a^3 + ab^2 + b^3$  is divisible by 11.

**Problem T-8.**

We call a positive integer  $n$  *amazing* if there exist positive integers  $a, b, c$  such that the equality

$$n = (b, c)(a, bc) + (c, a)(b, ca) + (a, b)(c, ab)$$

holds. Prove that there exist 2011 consecutive positive integers which are amazing. (By  $(m, n)$  we denote the greatest common divisor of positive integers  $m$  and  $n$ .)

*Time: 5 hours*

*Time for questions: 45 min*

*Each problem is worth 8 points.*

*The order of the problems does not depend on their difficulty.*