5th MIDDLE EUROPEAN MATHEMATICAL OLYMPIAD
VARAŽDIN 2011 CROATIA

language: English

5^{th} Middle European Mathematical Olympiad

Team Competition

4^{th} September 2011

Problem T-1.
Find all functions $f\colon \mathbb{R} \to \mathbb{R}$ such that the equality

$$y^2 f(x) + x^2 f(y) + xy = xyf(x+y) + x^2 + y^2$$

holds for all $x, y \in \mathbb{R}$, where $\mathbb{R}$ is the set of real numbers.

Problem T-2.
Let a, b, c be positive real numbers such that

$$\frac{a}{1+a} + \frac{b}{1+b} + \frac{c}{1+c} = 2.$$

Prove that

$$\frac{\sqrt{a}+\sqrt{b}+\sqrt{c}}{2} \geqslant \frac{1}{\sqrt{a}} + \frac{1}{\sqrt{b}} + \frac{1}{\sqrt{c}}.$$

Problem T-3.
For an integer $n \geq 3$, let $\mathcal{M}$ be the set $\{(x, y) \mid x, y \in \mathbb{Z},\ 1 \leq x \leq n, 1 \leq y \leq n\}$ of points in the plane. ($\mathbb{Z}$ is the set of integers.)

What is the maximum possible number of points in a subset $S \subseteq \mathcal{M}$ which does not contain three distinct points being the vertices of a right triangle?

Problem T-4.
Let $n \geq 3$ be an integer. At a MEMO-like competition, there are $3n$ participants, there are n languages spoken, and each participant speaks exactly three different languages.

Prove that at least $\left\lceil \frac{2n}{9} \right\rceil$ of the spoken languages can be chosen in such a way that no participant speaks more than two of the chosen languages.

($\lceil x \rceil$ is the smallest integer which is greater than or equal to x.)

Problem T-5.
Let $ABCDE$ be a convex pentagon with all five sides equal in length. The diagonals AD and EC meet in S with $\angle ASE = 60^\circ$. Prove that $ABCDE$ has a pair of parallel sides.

Problem T-6.
Let ABC be an acute triangle. Denote by B_0 and C_0 the feet of the altitudes from vertices B and C, respectively. Let X be a point inside the triangle ABC such that the line BX is tangent to the circumcircle of the triangle AXC_0 and the line CX is tangent to the circumcircle of the triangle AXB_0. Show that the line AX is perpendicular to BC.

Problem T-7.
Let A and B be disjoint nonempty sets with $A \cup B = \{1, 2, 3, \ldots, 10\}$. Show that there exist elements $a \in A$ and $b \in B$ such that the number $a^3 + ab^2 + b^3$ is divisible by 11.

Problem T-8.
We call a positive integer n *amazing* if there exist positive integers a, b, c such that the equality

$$n = (b, c)(a, bc) + (c, a)(b, ca) + (a, b)(c, ab)$$

holds. Prove that there exist 2011 consecutive positive integers which are amazing.
(By (m, n) we denote the greatest common divisor of positive integers m and n.)

Time: 5 hours
Time for questions: 45 min
Each problem is worth 8 points.
The order of the problems does not depend on their difficulty.