5th MIDDLE EUROPEAN MATHEMATICAL OLYMPIAD

VARAŽDIN 2011 CROATIA

# PROBLEMS AND SOLUTIONS

**5th Middle European Mathematical Olympiad**

**Varaždin, Croatia, September 2011**

### I 1 (Vjekoslav Kovač, Croatia)

Initially, only the integer 44 is written on a board. An integer $a$ on the board can be replaced with four pairwise different integers $a_1$, $a_2$, $a_3$, $a_4$ such that the arithmetic mean $\frac{1}{4}(a_1+a_2+a_3+a_4)$ of the four new integers is equal to the number $a$. In a step we simultaneously replace all the integers on the board in the above way. After 30 steps we end up with $n=4^{30}$ integers $b_1, b_2, \ldots, b_n$ on the board. Prove that

$$\frac{b_1^2+b_2^2+\cdots+b_n^2}{n} \geqslant 2011\,.$$

### First solution

Let us first prove an auxiliary statement.

*Lemma.* If $a_1, a_2, a_3, a_4$ are four different integers such that their average $a=(a_1+a_2+a_3+a_4)/4$ is also an integer, then

$$\frac{a_1^2+a_2^2+a_3^2+a_4^2}{4}-a^2 \geqslant \frac{5}{2}\,.$$

*Proof.* Note that the expression on the left hand side can be transformed as

$$\begin{aligned}
&\frac{a_1^2+a_2^2+a_3^2+a_4^2}{4}-a^2\\
&= \frac{a_1^2+a_2^2+a_3^2+a_4^2-8a^2+4a^2}{4}\\
&= \frac{a_1^2+a_2^2+a_3^2+a_4^2-2a(a_1+a_2+a_3+a_4)+4a^2}{4}\\
&= \frac{(a_1-a)^2+(a_2-a)^2+(a_3-a)^2+(a_4-a)^2}{4}\,.
\end{aligned}$$

Now, $a_1-a$, $a_2-a$, $a_3-a$, $a_4-a$ are four different integers that add up to 0. We claim that sum of their squares is at least 10. If none of these integers is 0, then that sum is at least $1^2+(-1)^2+2^2+(-2)^2=10$. On the other hand, if one of the integers is 0, than the remaining three cannot be only from the set $\{1,-1,2,-2\}$, because no three different elements of that set add up to 0. Therefore, the sum of their squares is at least $3^2+1^2+(-1)^2=11$. This completes the proof of the lemma.

Returning to the given problem, we denote by $S_k$ the average of squares of the numbers on the board after $k$ steps. More precisely,

$$S_k=\frac{b_{k,1}^2+b_{k,2}^2+\cdots+b_{k,4^k}^2}{4^k}\,,$$

where $b_{k,1}, b_{k,2}, \ldots, b_{k,4^k}$ are the numbers appearing on the board after the operation is performed $k$ times. Applying the above lemma to each of the numbers, adding up these inequalities, and dividing by $4^k$, we obtain $S_{k+1}-S_k \geqslant \frac{5}{2}$, so in particular

$$S_{30} \;\geqslant\; S_0+30\cdot\frac{5}{2} \;=\; 44^2+30\cdot\frac{5}{2} \;=\; 2011\,.$$

### Second solution (by Michał Zając, Poland)

Let $a_{0,1} = 44$ and let $a_{i,1}$, $a_{i,2}$, $\ldots$, $a_{i,4^i}$ be number written on the board after $i$ steps. In $(i+1)$-st step we replace the number $a_{i,k}$ with $a_{i+1,4k-3}$, $a_{i+1,4k-2}$, $a_{i+1,4k-1}$ and $a_{i+1,4k}$. We denote

$$S_i = \frac{\sum\limits_{j=1}^{4} a_{i,j}^2}{4^i}.$$

We want to prove that $S_{i+1} \geqslant S_i + 2.5$, with equality occuring when each number $a$ is replaced by $(a-2, a-1, a+1, a+2)$. For a given number $a$, let $(b_1, b_2, b_3, b_4)$ be an arbitrary quadruple of integers that satisfy the conditions that $b_1 + b_2 + b_3 + b_4 = 4a$ and $b_1 > b_2 > b_3 > b_4$. We will prove that $(b_1, b_2, b_3, b_4)$ majorizes $(a+2, a+1, a-1, a-2)$.

First we conclude that $b_1 \geqslant a + 2$, otherwise

$$b_1 + b_2 + b_3 + b_4 \leqslant (a+1) + a + (a-1) + (a-2) < 4a.$$

Next, it holds that $b_1 + b_2 \geqslant (a+2) + (a+1) = 2a + 3$.
Otherwise, it holds that $b_1 + b_2 \leqslant 2a + 2$ and thus $b_2 \leqslant a$, $b_3 \leqslant a - 1$ and $b_4 \leqslant a - 2$. This implies that $b_1 + b_2 + b_3 + b_4 \leqslant 4a - 1 < 4a$, which is false.

Finally, in order to prove that $b_1 + b_2 + b_3 \geqslant 3a + 2$, which is equivalent to $b_4 \leqslant a - 2$, we assume otherwise: $b_4 \geqslant a - 1$ and we arrive to contradiction in the same way as in the first case (in this case the sum is strictly bigger than $4a$). Thus, we have proved that $(b_1, b_2, b_3, b_4) \succ (a+2, a+1, a-1, a-2)$.

The function $f(x) = x^2$ is convex (because $f''(x) = 2 > 0$) and by Karamata inequality it holds that:

$$b_1^2 + b_2^2 + b_3^2 + b_4^2 \geqslant (a+2)^2 + (a+1)^2 + (a-1)^2 + (a-2)^2 = 4a^2 + 10.$$

Similar to first solution, we conclude that $S_{i+1} \geqslant S_i + 2.5$ and finally by inductive argument:

$$S_{30} \geqslant S_0 + 30 \cdot 2.5 = 2011.$$

## T 1 (Tonći Kokan, Croatia)

Find all functions $f\colon \mathbb{R} \to \mathbb{R}$ such that the equality

$$y^2 f(x) + x^2 f(y) + xy = xyf(x+y) + x^2 + y^2$$

holds for all $x, y \in \mathbb{R}$, where $\mathbb{R}$ is the set of real numbers.

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### First solution

Substituting $y = 0$ we find that $x^2 f(0) = x^2$ holds for all real numbers $x$ which implies $f(0) = 1$.

Let us introduce a new function $g : \mathbb{R} \to \mathbb{R}$ given by $g(x) = f(x) - 1$. Equation from the problem becomes

$$y^2 g(x) + x^2 g(y) = xy\, g(x+y), \tag{1}$$

while $g(0) = 0$.

Denoting $c = g(1)$ and introducing another function $h : \mathbb{R} \to \mathbb{R}$ defined by $h(x) = g(x) - cx$, we obviously get $h(0) = h(1) = 0$, whereas the equation that must be satisfied is now

$$y^2 h(x) + x^2 h(y) = xy\, h(x+y). \tag{2}$$

Substituting $x = y = 1$ in the last equation we get $h(2) = 0$, while another substitution $x = -1$, $y = 1$ gives $h(-1) = 0$.

Let us suppose that there exists a real number $y_0$ such that $h(y_0) \neq 0$.

Putting $x = 1$, $y = y_0 + 1$ in (2) we get:

$$h(y_0+1) = (y_0+1)h(y_0+2), \quad \text{or} \quad h(y_0+2) = \frac{h(y_0+1)}{y_0+1}. \tag{3}$$

On the other hand, substituting $x = 2$, $y = y_0$ in (2) gives

$$4h(y_0) = 2y_0 h(y_0+2), \quad \text{i.e.} \quad h(y_0+2) = \frac{2h(y_0)}{y_0}. \tag{4}$$

Finally, putting $x = 1$, $y = y_0$ in (2) leads to:

$$h(y_0) = y_0 h(y_0+1), \quad \text{or} \quad h(y_0+1) = \frac{h(y_0)}{y_0}. \tag{5}$$

From (3), (4) and (5) it follows that $y_0 = -\frac{1}{2}$. However, substituting $x = y = -\frac{1}{2}$ in (2) and using $h(-1) = 0$ we arrive at $h(-\frac{1}{2}) = 0$, which is a contradiction.

We conclude that $h(x) = 0$ holds for all $x \in \mathbb{R}$ and thus $f(x) = cx + 1$ is the only solution. We check that this really is the solution for every real number $c$.

### Second solution (by Matija Bašić, coordinator)

We define function $h$ as in the first solution of the problem. Hence, we have $h(0) = 0$, $h(1) = 0$,

$$y^2 h(x) + x^2 h(y) = xyh(x + y). \tag{$*$}$$

Substituting $y = x$ : $2x^2 h(x) = x^2 h(2x),\ \forall x,$ or $h(2x) = 2h(x)$ for all $x$.

Thus $h(2) = 0$.

Substituting $y = -x$ : $x^2 \left(h(x) + h(-x)\right) = x^2 h(0) = 0,\ \forall x,$ which gives

$$h(-x) = -h(x), \quad \forall x.$$

Thus $h(-1) = 0$.

Put $y = 1$ in $(*)$ : $h(x) + x^2 h(1) = xh(x + 1)$ i.e.

$$h(x) = xh(x + 1) \tag{1}$$

In (1) we change $x \to x + 1$

$$h(x + 1) = (x + 1)h(x + 2) \tag{2}$$

Put $y = 2$ in $(*)$ : $4h(x) + x^2 h(2) = 2xh(x + 2)$ i.e.

$$2h(x) = xh(x + 2) \tag{3}$$

Now we conclude

$$\begin{aligned} 2(x + 1)h(x) &= (3) = x(x + 1)h(x + 2) \\ &= (2) = xh(x + 1) \\ &= (1) = h(x) \end{aligned}$$

Therefore,

$$2(x + 1)h(x) = h(x), \quad \forall x$$

so $h(x) = 0$ or $2(x + 1) = 1$ for all $x$. Obviously, $h(x) = 0$ for all $x \neq -\frac{1}{2}$.

Moreover, $2h(-\frac{1}{2}) = h(2 \cdot (-\frac{1}{2})) = h(-1) = 0$ so $h(-\frac{1}{2}) = 0$ holds as well.

We have proved that $h(x) = 0$ for all $x \in \mathbb{R}$, hence, $g(x) = cx$, $f(x) = cx + 1$.

Direct check shows that $f(x) = cx + 1$ is the solution of the given functional equation for all $c \in \mathbb{R}$.

**Third solution (by Klemen Šivic, Slovenian leader)**

As in the first solution we obtain $f(0) = 1$ and we define $g(x) = f(x) - 1$. Then $g(0) = 0$ and

$$g(x+y) = \frac{y}{x}\, g(x) + \frac{x}{y}\, g(y) \quad \text{for } x, y \neq 0. \tag{1}$$

Therefore

$$g(x+y+z) = \frac{y+z}{x}\, g(x) + \frac{x}{y+z}\, g(y+z) = \frac{y+z}{x}\, g(x) + \frac{xy}{z(y+z)}\, g(z) + \frac{xz}{y(y+z)}\, g(y)$$

for all nonzero $x$, $y$ and $z$ such that $z \neq -y$. However, since the left side of the above equation is symmetric in $x$ and $z$, we obtain that

$$\frac{y+z}{x}\, g(x) + \frac{xy}{z(y+z)}\, g(z) + \frac{xz}{y(y+z)}\, g(y) = \frac{y+x}{z}\, g(z) + \frac{zy}{x(y+x)}\, g(x) + \frac{xz}{y(y+x)}\, g(y)$$

for all nonzero $x$, $y$ and $z$ such that $y \neq -x$ and $y \neq -z$. In this equation we set $y = z = 1$ and we obtain

$$\frac{2g(x)}{x} + x\, g(1) = \frac{1}{x(x+1)}\, g(x) + \left(x + 1 + \frac{x}{x+1}\right) g(1) \quad \text{for all } x \neq 0, -1,$$

i.e.

$$\frac{2x+1}{x(x+1)}\, g(x) = \frac{2x+1}{x+1}\, g(1) \quad \text{for all } x \neq 0, -1.$$

Therefore

$$g(x) = g(1)\, x \quad \text{for all}\, x \neq 0, -1, -\frac{1}{2}.$$

Clearly, the above equation holds also for $x = 0$. If we set $x = 1$ and $y = -1$ into the equation (1), we obtain $g(-1) - g(1)$, and if we set $x = y = -\frac{1}{2}$, then we obtain $-g(1) = g(-1) = 2g\left(-\frac{1}{2}\right)$, therefore $g\left(-\frac{1}{2}\right) = -\frac{g(1)}{2}$. Hence $g(x) = g(1)\, x$ for all $x \in \mathbb{R}$. $f(1) = a$ can be arbitrary, therefore all solutions are functions $g(x) = ax$, or equivalently, $f(x) = ax + 1$ for all $x \in \mathbb{R}$, where $a \in \mathbb{R}$ is arbitrary.

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**Fourth solution (by team Hungary)**

Similar to the first solution, we introduce the function $g(x)$ and prove that $g(0) = 0$ and $g(-x) = -g(x)$. Inserting $y = 1$ and $y = -1$ into the equation for $g(x)$ we ge:

$$g(x) + x^2 g(1) = x\, g(x+1), \tag{1}$$

$$g(x) + x^2 g(-1) = -x\, g(x-1). \tag{2}$$

Inserting $x + 1$ instead of $x$ into (2) we get:

$$g(x+1) + (x+1)^2 g(-1) = -(x+1)\, g(x). \tag{3}$$

From (2) and (3) we get represent $g(x+1)$ in two ways:

$$g(x+1) = \frac{g(x) + x^2 g(1)}{x} = -(x+1)\, g(x) - (x+1)^2 g(-1) \quad \text{for } x \neq 0.$$

Solving for $g(x)$ and using $g(-1) = -g(1)$ we get:

$$g(x)\left(x^2 + x + 1\right) = g(1)\, x \left(x^2 + x + 1\right).$$

Since $x^2 + x + 1 > 0$ for all $x \in \mathbb{R}$ we get $g(x) = g(1)\, x$ and $f(x) = cx + 1$. Direct check shows that this is, indeed, the solution of the given functional equation for all $c \in \mathbb{R}$.

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## T 2 (Kristina Ana Škreb, Croatia)

Let $a$, $b$, $c$ be positive real numbers such that

$$\frac{a}{1+a}+\frac{b}{1+b}+\frac{c}{1+c}=2.$$

Prove that

$$\frac{\sqrt{a}+\sqrt{b}+\sqrt{c}}{2}\geqslant\frac{1}{\sqrt{a}}+\frac{1}{\sqrt{b}}+\frac{1}{\sqrt{c}}.$$

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### First solution

Note that the condition of the problem is equivalent to

$$\frac{1}{1+a}+\frac{1}{1+b}+\frac{1}{1+c}=1. \tag{1}$$

We want to prove that

$$\begin{aligned}
&\frac{\sqrt{a}+\sqrt{b}+\sqrt{c}}{2}\geqslant\frac{1}{\sqrt{a}}+\frac{1}{\sqrt{b}}+\frac{1}{\sqrt{c}}\\
\iff\quad &\sqrt{a}+\sqrt{b}+\sqrt{c}\geqslant 2\left(\frac{1}{\sqrt{a}}+\frac{1}{\sqrt{b}}+\frac{1}{\sqrt{c}}\right)\\
\iff\quad &\left(\sqrt{a}+\frac{1}{\sqrt{a}}\right)+\left(\sqrt{b}+\frac{1}{\sqrt{b}}\right)+\left(\sqrt{c}+\frac{1}{\sqrt{c}}\right)\geqslant 3\left(\frac{1}{\sqrt{a}}+\frac{1}{\sqrt{b}}+\frac{1}{\sqrt{c}}\right)\\
\iff\quad &\frac{a+1}{\sqrt{a}}+\frac{b+1}{\sqrt{b}}+\frac{c+1}{\sqrt{c}}\geqslant 3\left(\frac{1}{\sqrt{a}}+\frac{1}{\sqrt{b}}+\frac{1}{\sqrt{c}}\right) \qquad (2)
\end{aligned}$$

From (1) we see that at most one of the numbers $a$, $b$, and $c$ can be strictly smaller than 1. (Otherwise, we would have $\frac{1}{1+a}+\frac{1}{1+b}+\frac{1}{1+c}>\frac{1}{2}+\frac{1}{2}=1$.)
Without loss of generality we can take $a\geqslant b\geqslant c$.

<u>Case 1.</u> $a\geqslant b\geqslant c\geqslant 1$
We have

$$\sqrt{a}\left(\sqrt{ab}-1\right)\geqslant\sqrt{b}\left(\sqrt{ab}-1\right)\quad\Longrightarrow\quad\frac{a+1}{\sqrt{a}}\geqslant\frac{b+1}{\sqrt{b}},$$

and also

$$\sqrt{b}\left(\sqrt{bc}-1\right)\geqslant\sqrt{c}\left(\sqrt{bc}-1\right)\quad\Longrightarrow\quad\frac{b+1}{\sqrt{b}}\geqslant\frac{c+1}{\sqrt{c}}.$$

<u>Case 2.</u> $a\geqslant b\geqslant 1$, and $c<1$
The same way as in Case 1, we get $\frac{a+1}{\sqrt{a}}\geqslant\frac{b+1}{\sqrt{b}}$.
Since $a$, $b$, and $c$ are positive numbers, (1) implies

$$\frac{1}{1+b}\leqslant 1-\frac{1}{1+c}=\frac{c}{1+c}\quad\Longrightarrow\quad bc\geqslant 1\quad\Longrightarrow\quad b\geqslant\frac{1}{c}.$$

And this gives

$$\sqrt{b}\left(\sqrt{\frac{b}{c}}-1\right)\geqslant\sqrt{\frac{1}{c}}\left(\sqrt{\frac{b}{c}}-1\right)\quad\Longrightarrow\quad\frac{b+1}{\sqrt{b}}\geqslant\frac{c+1}{\sqrt{c}}.$$

We have showed that

$$a \geqslant b \geqslant c \quad \Longrightarrow \quad \frac{a+1}{\sqrt{a}} \geqslant \frac{b+1}{\sqrt{b}} \geqslant \frac{c+1}{\sqrt{c}} \tag{3}$$

and

$$a \geqslant b \geqslant c \quad \Longrightarrow \quad \frac{1}{1+a} \leqslant \frac{1}{1+b} \leqslant \frac{1}{1+c} \tag{4}$$

hold.

Now (3), (4) and the Chebyshev inequality imply

$$\begin{aligned} \frac{a+1}{\sqrt{a}} + \frac{b+1}{\sqrt{b}} + \frac{c+1}{\sqrt{c}} &= \left(\frac{a+1}{\sqrt{a}} + \frac{b+1}{\sqrt{b}} + \frac{c+1}{\sqrt{c}}\right)\left(\frac{1}{1+a} + \frac{1}{1+b} + \frac{1}{1+c}\right) \\ &\geqslant 3\left(\frac{1}{\sqrt{a}} + \frac{1}{\sqrt{b}} + \frac{1}{\sqrt{c}}\right), \end{aligned}$$

which is exactly (2).

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**Second solution (by Klemen Šivic, Slovenian leader)**

We make a substitution $x = \frac{1}{a+1}, y = \frac{1}{b+1}, z = \frac{1}{c+1}$. The condition

$$\frac{1}{1+a} + \frac{1}{1+b} + \frac{1}{1+c} = 1$$

is then equivalent to

$$x + y + z = 1,$$

and the original variables can be expressed as $a = \frac{1}{x} - 1 = \frac{1-x}{x} = \frac{y+z}{x}$, $b = \frac{x+z}{y}$ and $c = \frac{x+y}{z}$. The inequality

$$\frac{\sqrt{a} + \sqrt{b} + \sqrt{c}}{2} \geqslant \frac{1}{\sqrt{a}} + \frac{1}{\sqrt{b}} + \frac{1}{\sqrt{c}}$$

is then equivalent to

$$\sqrt{\frac{x+y}{2z}} + \sqrt{\frac{y+z}{2x}} + \sqrt{\frac{z+x}{2y}} \geqslant \sqrt{\frac{2x}{y+z}} + \sqrt{\frac{2y}{z+x}} + \sqrt{\frac{2z}{y+x}}.$$

We will prove that this inequality holds for all positive numbers $x$, $y$ and $z$.

We make a substitution $p = x + y, q = y + z, r = z + x$. Then $p$, $q$ and $r$ are sides of a triangle and we have to prove that

$$\sqrt{\frac{p}{q+r-p}} + \sqrt{\frac{q}{r+p-q}} + \sqrt{\frac{r}{p+q-r}} \geqslant \sqrt{\frac{p+q-r}{r}} + \sqrt{\frac{q+r-p}{p}} + \sqrt{\frac{r+p-q}{q}}. \tag{1}$$

Since $p$, $q$ and $r$ are sides of a triangle, we can write $p = 2R\sin\alpha$, $q = 2R\sin\beta$ and $r = 2R\sin\gamma$, where $R$ is the circumradius and $\alpha$, $\beta$ and $\gamma$ angles of the triangle with sides $p$, $q$ and $r$. Then

$$\begin{aligned} \sqrt{\frac{p}{q+r-p}} &= \sqrt{\frac{\sin\alpha}{\sin\beta + \sin\gamma - \sin\alpha}} = \sqrt{\frac{\sin(\beta+\gamma)}{\sin\beta + \sin\gamma - \sin(\beta+\gamma)}} = \\ &= \sqrt{\frac{2\sin\frac{\beta+\gamma}{2}\cos\frac{\beta+\gamma}{2}}{2\sin\frac{\beta+\gamma}{2}(\cos\frac{\beta-\gamma}{2} - \cos\frac{\beta+\gamma}{2})}} = \sqrt{\frac{\sin\frac{\alpha}{2}}{2\sin\frac{\beta}{2}\sin\frac{\gamma}{2}}}. \end{aligned}$$

Similarly we compute the other terms in (1), therefore (1) is equivalent to

$$\sqrt{\frac{\sin\frac{\alpha}{2}}{2\sin\frac{\beta}{2}\sin\frac{\gamma}{2}}}+\sqrt{\frac{\sin\frac{\beta}{2}}{2\sin\frac{\gamma}{2}\sin\frac{\alpha}{2}}}+\sqrt{\frac{\sin\frac{\gamma}{2}}{2\sin\frac{\alpha}{2}\sin\frac{\beta}{2}}}$$
$$\geqslant\sqrt{\frac{2\sin\frac{\alpha}{2}\sin\frac{\beta}{2}}{\sin\frac{\gamma}{2}}}+\sqrt{\frac{2\sin\frac{\beta}{2}\sin\frac{\gamma}{2}}{\sin\frac{\alpha}{2}}}+\sqrt{\frac{2\sin\frac{\gamma}{2}\sin\frac{\alpha}{2}}{\sin\frac{\beta}{2}}},$$

or equivalently, to

$$\sin\frac{\alpha}{2}+\sin\frac{\beta}{2}+\sin\frac{\gamma}{2}\geqslant 2\left(\sin\frac{\alpha}{2}\sin\frac{\beta}{2}+\sin\frac{\alpha}{2}\sin\frac{\gamma}{2}+\sin\frac{\beta}{2}\sin\frac{\gamma}{2}\right)$$
$$=\left(\sin\frac{\alpha}{2}+\sin\frac{\beta}{2}+\sin\frac{\gamma}{2})^2-(\sin^2\frac{\alpha}{2}+\sin^2\frac{\beta}{2}+\sin^2\frac{\gamma}{2}\right).$$

Since $\sin x$ is concave function on $(0,\pi)$, Jensen's inequality implies that

$$\sin\frac{\alpha}{2}+\sin\frac{\beta}{2}+\sin\frac{\gamma}{2}\leqslant 3\sin\frac{\alpha+\beta+\gamma}{6}=3\sin\frac{\pi}{6}=\frac{3}{2}.$$

Therefore

$$\sin\frac{\alpha}{2}+\sin\frac{\beta}{2}+\sin\frac{\gamma}{2}\geqslant\frac{2}{3}\left(\sin\frac{\alpha}{2}+\sin\frac{\beta}{2}+\sin\frac{\gamma}{2}\right)^2$$
$$\geqslant\left(\sin\frac{\alpha}{2}+\sin\frac{\beta}{2}+\sin\frac{\gamma}{2}\right)^2-\left(\sin^2\frac{\alpha}{2}+\sin^2\frac{\beta}{2}+\sin^2\frac{\gamma}{2}\right),$$

where at the end we used the arithmetic-quadratic mean. Therefore the inequality is proved.

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**Third solution (by team Croatia)**

Let $a=2x, b=2y, c=2z$. Then our condition is equivalent to :

$$\frac{x}{1+2x}+\frac{y}{1+2y}+\frac{z}{1+2z}=1\quad\iff\quad\frac{1}{1+2x}+\frac{1}{1+2y}+\frac{1}{1+2z}=2.$$

and we need to prove that

$$\sqrt{x}+\sqrt{y}+\sqrt{z}\geqslant\frac{1}{\sqrt{x}}+\frac{1}{\sqrt{y}}+\frac{1}{\sqrt{z}},$$

which is equivalent to :

$$\sum_{cyc}\frac{x-1}{\sqrt{x}}\geqslant 0\quad\iff\quad\sum_{cyc}\frac{x-1}{2x+1}\cdot\frac{2x+1}{\sqrt{x}}\geqslant 0.$$

Since this inequality is symmetric, we can assume $x\geqslant y\geqslant z$. We prove that then:

$$\frac{x-1}{2x+1}\geqslant\frac{y-1}{2y+1}\geqslant\frac{z-1}{2z+1}\tag{1}$$

and

$$\frac{2x+1}{\sqrt{x}}\geqslant\frac{2y+1}{\sqrt{y}}\geqslant\frac{2z+1}{\sqrt{z}}.\tag{2}$$

In order to prove (1) we note that:

$$\frac{x-1}{2x+1}\geqslant\frac{y-1}{2y+1}\quad\iff\quad 3x\geqslant 3y,$$

which holds. The same argument holds for $y$ and $z$.

In order to prove (2) we factor the inequality in the following equivalent way:

$$(\sqrt{x}-\sqrt{y})(2\sqrt{xy}-1) \geqslant 0.$$

By the assumption, $\sqrt{x}-\sqrt{y} \geqslant 0$ thus we need to prove that $2\sqrt{xy}-1 \geqslant 0$. Assume the opposite, ie. that $4xy < 1$. Then:

$$\frac{1}{1+2x}+\frac{1}{1+2y}=\frac{2(1+x+y)}{1+2(x+y)+4xy}=1+\frac{1-4xy}{(1+2x)(1+2y)}>1,$$

which contradicts the condition.

We have proven that triplets

$$\left(\frac{x-1}{2x+1},\frac{y-1}{2y+1},\frac{z-1}{2z+1}\right) \quad \text{and} \quad \left(\frac{2x+1}{\sqrt{x}},\frac{2y+1}{\sqrt{y}},\frac{2z+1}{\sqrt{z}}\right)$$

are ordered in the same way thus by Chebyshev inequality we have:

$$\sum_{cyc}\left(\frac{x-1}{2x+1}\cdot\frac{2x+1}{\sqrt{x}}\right) \geqslant \frac{1}{3}\sum_{cyc}\frac{x-1}{2x+1}\cdot\sum_{cyc}\frac{2x+1}{\sqrt{x}}=0.$$

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## I 2 (Tomislav Pejković, Croatia)

Let $n \geqslant 3$ be an integer. John and Mary play the following game: First John labels the sides of a regular $n$-gon with the numbers $1, 2, \ldots, n$ in whatever order he wants, using each number exactly once. Then Mary divides this $n$-gon into triangles by drawing $n-3$ diagonals which do not intersect each other inside the $n$-gon. All these diagonals are labeled with number 1. Into each of the triangles the product of the numbers on its sides is written. Let $S$ be the sum of those $n-2$ products.

Determine the value of $S$ if Mary wants the number $S$ to be as small as possible and John wants $S$ to be as large as possible and if they both make the best possible choices.

## Solution (by Rudi Mrazović, coordinator)

For $n = 3$ the answer is 6. Suppose $n \geqslant 4$. It is obvious that in each triangulation there are at least two triangles that share two sides with the polygon. We will prove that it is always best for Mary to choose a triangulation for which there is no more than two triangles of this kind.

We call a triangle in a triangulation *bad* if all of its sides are diagonals of the polygon. First we prove that Mary can choose an optimal triangulation that contains no bad triangles. Assume on the contrary that every optimal triangulation contains a bad triangle. For an optimal triangulation $\mathcal{T}$ let $d(\mathcal{T})$ be the length of the smallest side of all bad triangles in $\mathcal{T}$. Among all optimal triangulations with minimal number of bad triangles let $\mathcal{T}_0$ be such that $d(\mathcal{T}_0)$ is minimal.

Consider a bad triangle $ABC$ in $\mathcal{T}_0$ such that $|AB| = d(\mathcal{T}_0)$. Let $ABD$ be the other triangle of $\mathcal{T}_0$ that contains $\overline{AB}$ as one of its sides. Since $D$ lies on the arc $\widehat{AB}$ of the circumcircle of $ABC$ that does not contain $C$ and $\sphericalangle ACB$ is acute, we have $|AD| < |AB|$ and $|BD| < |AB|$.

Let $\mathcal{T}_1$ be the triangulation obtained from $\mathcal{T}_0$ by replacing $\overline{AB}$ with $\overline{CD}$. If the sides $\overline{AD}$ and $\overline{BD}$ have labels $a$ and $b$ respectively, then

$$S(\mathcal{T}_1) - S(\mathcal{T}_0) = a + b - ab - 1 = -(a-1)(b-1) \leqslant 0.$$

Because $\mathcal{T}_0$ is optimal triangulation, we conclude that $\mathcal{T}_1$ is also optimal. Since $\mathcal{T}_0$ has the minimal number of bad triangles at least one of the segments $\overline{AD}$ and $\overline{BD}$ should be a diagonal, but then $d(\mathcal{T}_1)$ is less than $d(\mathcal{T}_0)$ what is a contradiction.

Now that we know that Mary can choose an optimal triangulation that contains no bad triangles, we easily conclude that in a such triangulation there are exactly two triangles that share two sides with the polygon. If we denote by $x_1$ (respectively $x_2$) the number of triangles that have exactly one (respectively two) of their sides being the sides of the polygon, then $x_1 + x_2 = n - 2$ and $x_1 + 2x_2 = n$, so $x_2 = 2$.

Mary's strategy is to choose these two triangles so that the side of the polygon labeled with 1 is contained in one of these triangles and the side labeled with 2 is contained in the other.

By this strategy Mary makes sure that

$$
\begin{aligned}
S &\leqslant \max\left\{\frac{n(n+1)}{2} - (1+2+n+n-1) + 1\cdot n + 2\cdot(n-1),\right.\\
&\qquad\left.\frac{n(n+1)}{2} - (1+2+n+n-1) + 1\cdot(n-1) + 2\cdot n\right\}\\
&= \frac{n^2+3n-6}{2}.
\end{aligned}
$$

On the other hand, John can force Mary to achieve exactly this bound by labeling the sides of the polygon in the following order

$$1,\ n-1,\ 4,\ n-3,\ 5,\ \ldots,\ n-2,\ 3,\ n,\ 2.$$

Thus, the answer to our problem is $S = \dfrac{n^2+3n-6}{2}$, for each $n \geqslant 3$.

---

**T 3 (Viktor Harangi, Hungary)**

For an integer $n \geqslant 3$, let $\mathcal{M}$ be the set $\{(x, y) \mid x, y \in \mathbb{Z},\ 1 \leqslant x \leqslant n, 1 \leqslant y \leqslant n\}$ of points in the plane. ($\mathbb{Z}$ is the set of integers.)

What is the maximum possible number of points in a subset $S \subseteq \mathcal{M}$ which does not contain three distinct points being the vertices of a right triangle?

---

**Solution**

We will prove that the maximal cardinality of $S$ is $2n - 2$.

The set

$$S = \{1\} \times \{2, \ldots, n\} \cup \{2, \ldots, n\} \times \{1\}$$

has cardinality $2n - 2$ and it does not contain three distinct points that form a right triangle.

We will show that any subset $S \subset \mathcal{M}$ which does not contain three distinct points that form a right triangle can have at most $2n - 2$ points. For such set $S$ consider its subsets:

- $S_x$ consists of those points $P = (x, y)$ in $S$ that have *unique $x$ coordinate*, that is, there exists no $y' \neq y$ such that $(x, y') \in S$.
- $S_y$ consists of those points $P = (x, y)$ in $S$ that have *unique $y$ coordinate*, that is, there exists no $x' \neq x$ such that $(x', y) \in S$.

We claim that $S = S_x \cup S_y$. We prove this by contradiction. Assume ther exists a point $P \in S \setminus (S_x \cup S_y)$. Since $P \notin S_x$, there exists $P_x \neq P$ in $S$ with the same $x$ coordinate as $P$. Similarly, there exists $P_y \neq P$ in $S$ with the same $y$ coordinate as $P$. Hence $P, P_x, P_y \in S$ and $\sphericalangle P_x P P_y = 90^\circ$, a contradiction.

Clearly, $|S_x| \leqslant n$, and if $|S_x| = n$, then $S = S_x$. The same holds for $S_y$. So, $|S| = n$ or $|S_x|, |S_y| \leqslant n - 1$ and $|S| \leqslant |S_x| + |S_y| \leqslant 2n - 2$. It follows that the cardinality of $S$ is at most $\max(n, 2n - 2) = 2n - 2$.

---

## T 4 (Vjekoslav Kovač, Croatia)

Let $n \geqslant 3$ be an integer. At a MEMO-like competition, there are $3n$ participants, there are $n$ languages spoken, and each participant speaks exactly three different languages.

Prove that at least $\left\lceil \dfrac{2n}{9} \right\rceil$ of the spoken languages can be chosen in such a way that no participant speaks more than two of the chosen languages.

($\lceil x \rceil$ is the smallest integer which is greater than or equal to $x$.)

### First solution

Consider the classifications of the set of $n$ available languages into **easy**, **medium**, and **hard** languages. There are $3^n$ possible classifications in total and we denote by $S$ the set of all possible classifications. For each classification $s \in S$, let $A(s)$ be the number of easy languages and let $B(s)$ be the number of students who speak 3 easy languages.

If we add up quantities $A(s)$ over all possible classifications $s \in S$, the resulting sum will be $\sum_{s \in S} A(s) = n3^{n-1}$. In order to verify that, we realize that the result should be the same for medium and hard languages too, but all three of these sums add up to

$$3 \sum_{s \in S} A(s) \;=\; \text{number of classifications} \;\times\; \text{number of languages} \;=\; 3^n \cdot n\,.$$

On the other hand, we use double counting to compute the sum of quantities $B(s)$ over all possible classifications $s \in S$.

For each student there are $3^{n-3}$ classifications for which he speaks 3 easy languages, as we only have the choice to classify each of the $n-3$ languages that the student does not speak. In two ways, we count the cardinality of the set

$$\{(X, s) : \text{for a classification } s \text{ student } X \text{ speaks 3 easy languages}\}$$

to get the identity

$$\sum_{s \in S} B(s) \;=\; 3n \cdot 3^{n-3} \;=\; n3^{n-2}\,.$$

We claim that there exists a classification $s \in S$ such that $A(s) - B(s) \geqslant \frac{2n}{9}$. If we assume on the contrary that $A(s) - B(s) < \frac{2n}{9}$ for all classifications $s \in S$, then summing over all $3^n$ of them would give

$$n3^{n-1} - n3^{n-2} \;=\; \sum_{s \in S} A(s) - \sum_{s \in S} B(s) \;<\; 3^n \cdot \frac{2n}{9}\,,$$

i.e. $2n3^{n-2} < 2n3^{n-2}$, which is a contradiction.

Let us consider any classification $s \in S$ of languages satisfying $A(s) - B(s) \geqslant \frac{2n}{9}$. We can first choose all $A(s)$ easy languages. Then we find all $B(s)$ students who can speak 3 of these languages, and for each of them we remove one of the languages the student speaks. This leaves us with a choice of at least $\frac{2n}{9}$ languages.

**Remark:** Classification of languages simply as **easy** or **hard** would not give the desired bound. It would lead to a choice of at least $\frac{n}{8}$ languages only. Taking more than three language classes would not be a better strategy either.

**Solution (by Rudi Mrazović, coordinator)**

In this proof we will use probabilistic method. Let $p \in [0,1]$. For each language, suppose we choose it with probability $p$ and we make these decisions independently. [1] Let $A$ be the number of chosen languages (i.e. the number of 1s in $\omega$) and $B$ the number of students whose all three languages are among chosen ones. Lets calculate the expectations [2] of these random variables.

$$\begin{aligned}\mathbb{E}A &= \mathbb{E}\left[\sum_{\text{language } l} \mathbf{1}_{\text{we have chosen the language } l}\right] = \sum_{\text{language } l} \mathbb{E}\left[\mathbf{1}_{\text{we have chosen the language } l}\right] \\ &= \sum_{\text{language } l} \mathbb{P}\left(\text{we have chosen the language } l\right) = np.\end{aligned}$$

$$\begin{aligned}\mathbb{E}B &= \mathbb{E}\left[\sum_{\text{student } s} \mathbf{1}_{\text{student's } s \text{ languages are all chosen}}\right] = \sum_{\text{student } s} \mathbb{E}\left[\mathbf{1}_{\text{student's } s \text{ languages are all chosen}}\right] \\ &= \sum_{\text{student } s} \mathbb{P}\left(\text{student's } s \text{ languages are all chosen}\right) = 3np^3.\end{aligned}$$

We will use the following obvious (and easily proved inequality). For arbitrary random variable $X$ we have

$$\mathbb{P}(X \geqslant \mathbb{E}X) > 0.$$

For $X = A - B$ we get

$$\mathbb{P}(A - B \geqslant np - 3np^3) > 0.$$

In this way we have proved that there is a choosing of languages such that $A - B \geqslant np - 3np^3$. For this choosing for each student that speaks three chosen languages remove one of them. In the end we are left with at least $A - B$ (and thus $np - 3np^3$) languages that do the job. Taking $p = \frac{1}{3}$ we get what we need, i.e. we can choose at least $\lceil \frac{2n}{9} \rceil$ such that no student speaks more than two of them.

---

**Alternative approach (based on the solution by team Poland)**

We choose $\lceil \frac{n}{3} \rceil$ languages uniformly and randomly. Similarly to the previous probabilistic solution we show that with positive probability the number of students that speak three of the chosen languages is less or equal to $\lfloor \frac{n}{9} \rfloor$. Again, use the same trick of removing some of the languages to obtain at least $\lceil \frac{2n}{9} \rceil$ of them such that no student speaks three of them.

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[1] Formally, we consider probability space $(\{0,1\}^n, \mathcal{P}(\{0,1\}^n), \mathbb{P})$ where

$$\mathbb{P}(\omega) = p^{k(\omega)}(1-p)^{n-k(\omega)}, \quad \text{for each } \omega \in \{0,1\}^n$$

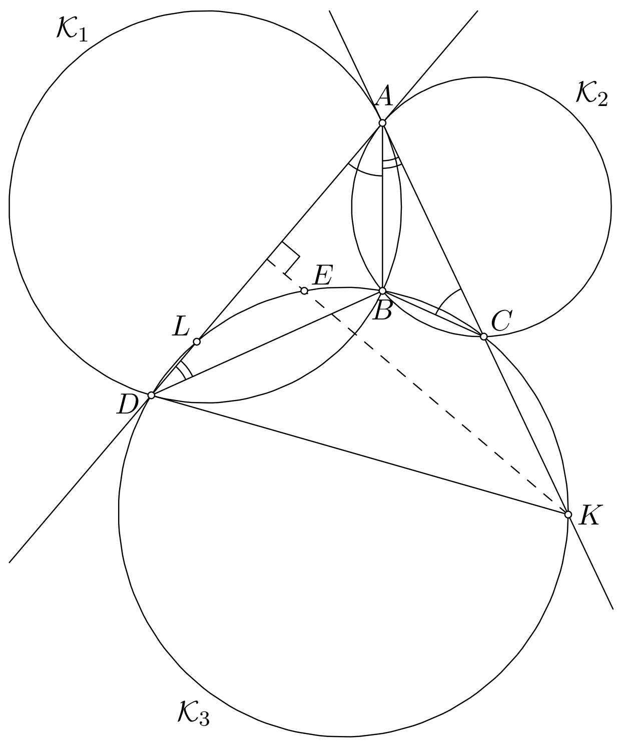
where $k(\omega)$ is the number of 1s in $\omega$.

[2] The expectation of integer random variable $X$ is the number $\mathbb{E}X = \sum_{k=0}^{n} k\mathbb{P}(X = k)$.

### I 3 (Nik Stopar, Slovenia)

In a plane the circles $\mathcal{K}_1$ and $\mathcal{K}_2$ with centers $I_1$ and $I_2$, respectively, intersect in two points $A$ and $B$. Assume that $\sphericalangle I_1AI_2$ is obtuse. The tangent to $\mathcal{K}_1$ in $A$ intersects $\mathcal{K}_2$ again in $C$ and the tangent to $\mathcal{K}_2$ in $A$ intersects $\mathcal{K}_1$ again in $D$. Let $\mathcal{K}_3$ be the circumcircle of the triangle $BCD$. Let $E$ be the midpoint of that arc $CD$ of $\mathcal{K}_3$ that contains $B$. The lines $AC$ and $AD$ intersect $\mathcal{K}_3$ again in $K$ and $L$, respectively. Prove that the line $AE$ is perpendicular to $KL$.

### First solution (by Tomislav Pejković, coordinator)




Since $AD$ is tangent to $\mathcal{K}_2$, it follows that $\sphericalangle ACB = \sphericalangle DAB$. Similarly, $\sphericalangle ADB = \sphericalangle BAC$.

From this we have $\sphericalangle DBC = (\sphericalangle ADB + \sphericalangle DAB) + (\sphericalangle BAC + \sphericalangle ACB) = 2(\sphericalangle DAB + \sphericalangle BAC)$, hence

$$\sphericalangle DBC = 2\sphericalangle DAC.$$

By $\widehat{XY}$ we denote the angle $\sphericalangle XZY$ where $Z$ is a point on the circle $\mathcal{K}_3$ such that $X, Y, Z$ are ordered counterclockwise.

Since $E$ is the midpoint of the arc $CD$ and the points $C, E, D, K$ are concyclic we have

$$\sphericalangle AKE = \frac{1}{2}\widehat{CD} = \frac{1}{2}(180^\circ - \widehat{DC}) = \frac{1}{2}(180^\circ - \sphericalangle CBD) = 90^\circ - \sphericalangle DAC.$$

This means that $KE$ and $AL$ are perpendicular.

Analogously, $LE$ and $AK$ are perpendicular and $E$ is the orthocenter of the triangle $AKL$. Hence $AE$ and $KL$ are perpendicular.

**Remark:** We use the notation $\widehat{CD}$ and $\widehat{DC}$ because it provides a convenient way of writing the solution in all cases regardless of the mutual position of the points $A, D, L, C, K$.

---

**Second solution**

Since $AD$ is tangent to $\mathcal{K}_2$, it follows that $\sphericalangle ACB = \sphericalangle DAB$. Similarly, $\sphericalangle ADB = \sphericalangle BAC$.

From this we have $\sphericalangle DBC = (\sphericalangle ADB + \sphericalangle DAB) + (\sphericalangle BAC + \sphericalangle ACB) = 2(\sphericalangle DAB + \sphericalangle BAC)$, hence

$$\sphericalangle DEC = \sphericalangle DBC = 2\sphericalangle DAC.$$

Since $|ED| = |EC|$, the point $E$ is the circumcenter of $ACD$. Therefore $|EC| = |EA| = |ED|$.

Because the points $C, B, D, K$ are concyclic we have $\sphericalangle KDB = \sphericalangle ACB$. From this and the first arguments of this solution we have that $|DK| = |AK|$. Since we proved $|EA| = |ED|$, we conclude that the line $KE$ is the bisector of the segment $AD$ and therefore perpendicular to it.

Analogously, $LE$ and $AK$ are perpendicular and $E$ is the orthocenter of the triangle $AKL$. Hence $AE$ and $KL$ are perpendicular.

**Remark:** The identity $\sphericalangle KDB = \sphericalangle ACB$ holds in all cases regardless of the mutual position of the points $A, D, L, C, K$.

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**Third solution (by Karol Kaszuba, Poland)**

Let us apply inversion with respect to a circle with the center $A$ and radius $r$. Denote the image of point $X$ with $X'$. From the assumptions of the problem and well known facts about the inversion directly follows that $AD'B'C'$ is a parallelogram.

From the definition of the image of the point by inversion we have

$$|E'C'| = |EC|\frac{r^2}{|AE||AC|}, \quad |E'D'| = |ED|\frac{r^2}{|AE||AD|}.$$

Dividing these two identities and using that $E$ is the midpoint of the arc $\widehat{CD}$ we obtain

$$\frac{|E'C'|}{|E'D'|} = \frac{|EC|}{|ED|} \cdot \frac{|AD|}{|AC|} = \frac{|AD|}{|AC|} = \frac{|AC'|}{|AD'|} = \frac{|D'B'|}{|C'B'|}.$$

We consider all points $X$ with the property

$$\frac{|XC'|}{|XD'|} = \frac{|D'B'|}{|C'B'|}.$$

These points form the Apollonius circle and hence there are exactly two such points intersecting the image of $\mathcal{K}_3$, each on different arc $\widehat{C'D'}$. One of these is the point $E'$. Since the point symmetric to $B'$ with respect to the line $C'D'$ also lies on the same arc $\widehat{C'D'}$ as $E'$ and lies on the mentioned Apollonius circle we conclude that $E'$ is symmetric to $B'$.

This implies $|E'C'| = |B'D'| = |C'A'|$ (first equality holds because of the symmetry the second because $AD'B'C'$ is a parallelogram) and similarly $|E'D'| = |D'A'|$. Hence $AC'E'D'$ is a deltoid so $AE' \perp C'D'$. This means that $AE'$ contains the orthocenter of the triangle $AC'D'$. It is well know that the orthocenter and circumcenter are isogonal conjugates (lying on the lines which are symmetric with respect to the angle bisector). On the other hand triangles $AC'D'$ and $AL'K'$ are inversely similar, so the circumcenter of $AK'L'$ lies on the same line through $A$ as the orthocenter of $AC'D'$.

All of this shows that $AE'$ pass through the circumcenter of $AK'L'$, so $AE$ is perpendicular to $KL$.

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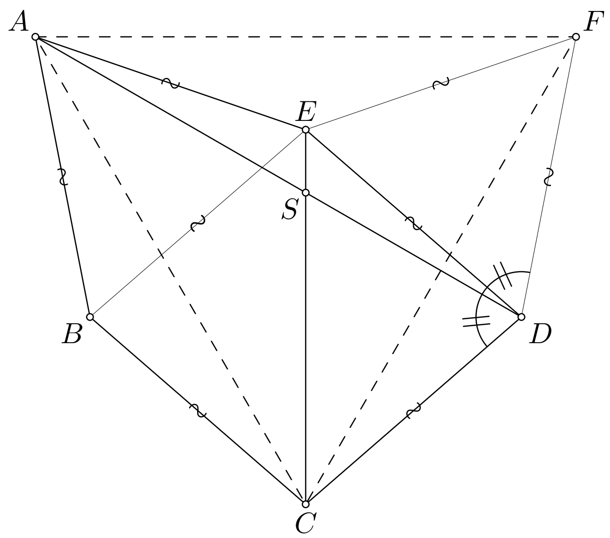
### T 5 (Michal Szabados, Slovakia)

Let $ABCDE$ be a convex pentagon with all five sides equal in length. The diagonals $AD$ and $EC$ meet in $S$ with $\sphericalangle ASE = 60°$. Prove that $ABCDE$ has a pair of parallel sides.

### First solution

Let $F$ be such that $DEF$ is an equilateral triangle and the points $B$ and $F$ lay in the opposite half-planes determined by $DE$. Denote $\sphericalangle DAE = \alpha$. Then $\sphericalangle ADE = \alpha$.




Since $\sphericalangle ESD = 120°$ , we have $\sphericalangle DEC = 60° - \alpha$. Then $\sphericalangle SCD = \sphericalangle ECD$ and

$$\sphericalangle ADC = \sphericalangle SDC = 180° - \sphericalangle SCD - \sphericalangle DSC = 60° + \alpha.$$

Obviously $\sphericalangle ADF = 60° + \alpha$ and because $|FD| = |CD|$ we conclude that $ADF \simeq ADC$. Similarly, $\sphericalangle AEC = \sphericalangle FEC = 120° - \alpha$, so $ACE \cong FCE$.

From these two pairs of equal triangles we conclude $|AF| = |AC| = |FC|$, so both triangles $DEF$ and $ACF$ are equilateral.

If $E$ lies on the line $AF$ or $D$ lies on the line $FC$ then $|AC| = 2|ED| = |AB| + |BC|$ and $B$ lies on $AC$, which is not possible. Therefore exactly one of the points $D$ and $E$ lays inside the triangle $ACF$. Without loss of generality, let it be the point $E$.

The triangles $AEF$ and $ABC$ have their corresponding sides equal therefore $AEF \cong ABC$ and this yields $60° = \sphericalangle FAC = \sphericalangle EAB$, so $|EB| = |AB|$. Hence $BCDE$ is a rhombus, i. e., $ED \parallel BC$.

---

**Second solution (by Matija Bašić, coordinator)**

Let $\alpha$ be as in the first solution. In the same way we prove $\sphericalangle AEC = 120^\circ - \alpha$ and $\sphericalangle CED = 60^\circ - \alpha$. Let $F$ be the symmetric image of $A$ with respect to $CE$. We get $\sphericalangle DEF = 120^\circ - \alpha - (60^\circ - \alpha) = 60^\circ$. Since $|DE| = |AE| = |EF|$, triangle $DEF$ is equilateral.

Because $|AB| = |BC| = |DF| = |CD|$ the triangles $ABC$ and $CDF$ are congruent.

If the point $D$ is outside the triangle $ACF$ then this implies that $B$ and $D$ are symmetric with respect to $CE$, so $|BE| = |DE|$. Hence $BCDE$ is a rhombus and $DE \| BC$.

If the point $D$ is inside the triangle $ACF$ then the point $E$ is outside that triangle and we see in the similar way that $F$ and $C$ are symmetric with respect to $AD$ and also $B$ and $E$ are symmetric with respect to $AD$. Hence $|BD| = |DE|$ and $ABDE$ is a rhombus, so $DE \parallel AB$.

---

**Third solution (by Gerd Baron, Austrian leader)**

Define the point $B'$ such that $B'CDE$ is rhombus.

If the pentagon $AB'CDE$ is convex, denote $\sphericalangle EAD = \sphericalangle EDA = \alpha$. Similarly to other solution we have $\sphericalangle AEB' = \sphericalangle AED - \sphericalangle B'EC - \sphericalangle CED = 180^\circ - 2\alpha - (60^\circ - \alpha) - (60^\circ - \alpha) = 60^\circ$.

Since $|AE| = |B'E|$, we conclude that $AB'E$ is equilateral.

Points $B$ and $B'$ are on the same side of the line $AC$, so we conclude that $B = B'$, so $DE \parallel AB$.

If the pentagon $AB'CDE$ is not convex, denote the intersection of $B'E$ and $AD$ by $F$ and $\sphericalangle DEC = \sphericalangle DCE = \beta$. Similarly to other solutions we have $AEB' = 180^\circ - \sphericalangle EAF - \sphericalangle EFA = 180^\circ - \sphericalangle EDA - (\sphericalangle FEC + \sphericalangle FSE) = 180^\circ - (60^\circ - \beta) - (\beta + 60^\circ) = 60^\circ$.

Since $|AE| = |B'E|$, we conclude that $AB'E$ is equilateral.

Let $B''$ be the symmetric image of $B'$ with respect to $AC$. Then $AB''CB'$ is a rhombus and $B = B''$, so we conclude $B'C \parallel AB''$ and hence $DE \parallel AB$.

### Fourth solution (by team Slovakia)

Denote $\sphericalangle DEC = \sphericalangle DCE = \alpha$ and suppose that all five sides of the pentagon have length $a$. As in the previous solutions we see that $\sphericalangle SEA = 60^\circ + \alpha, \sphericalangle SDC = 120^\circ - \alpha$. Applying the law of sines to the triangles $ASE$ and $CSD$ implies

$$|SA| = \frac{a\sin(60^\circ + \alpha)}{\sin 60^\circ} = \frac{a\sin(120^\circ - \alpha)}{\sin 60^\circ} = |SC|.$$

The triangle $ASC$ is isosceles and $\sphericalangle ACS = \sphericalangle CAS = 30^\circ$ and we have $|AC| = \sqrt{3} \cdot |AS|$.

The law of cosines applied to the triangle $ABC$ gives

$$a^2 = a^2 + 3|AS|^2 - 2\sqrt{3}a \cdot |AS| \cdot \cos(\sphericalangle ACB)$$

from where we get $\cos(\sphericalangle ACB) = \dfrac{3|AS|}{2\sqrt{3}a} = \sin(60^\circ + \alpha) = \cos(30^\circ - \alpha)$.

Since $0 < \sphericalangle ACB < 90^\circ$ we have two possibilities.

The first possibility is that $\sphericalangle ACB = 30^\circ - \alpha$, so $\sphericalangle BCE = \alpha = \sphericalangle CED$ and hence $BC \parallel ED$.

The second possibility is that $\sphericalangle ACB = \alpha - 30^\circ$, so $\sphericalangle BAD = 60^\circ - \alpha = \sphericalangle ADE$ and hence $AB \parallel ED$.

---

### Fifth solution (by team Germany)

We construct a point $Q$ on the line $SE$ such that $ASQ$ is the equilateral triangle. As in the previous solutions it is easily seen that $\sphericalangle EAQ = \sphericalangle DCS$ and since $\sphericalangle AQS = 60^\circ = \sphericalangle CSD$ and $|AE| = |DC|$ we have that the triangle $AEQ$ and $SCD$ are congruent, so $|AS| = |AQ| = |CS|$.

This shows that the quadrilateral $ABCS$ is a deltoid, so $\sphericalangle ASB = \sphericalangle BSC = 60^\circ$ and the point $S$ is the Fermat's point of the triangle $BDE$.

Let point $X$ be such that $BEX$ is equilateral and that $S$ and $X$ lie on different sides of the line $EB$. It is well know that the property of the Fermat's point $S$ is that $X, S$ and $D$ are collinear. Also, since $|BX| = |EX|$, $X$ lies on the bisector of the segment $\overline{BE}$.

We have two cases. In the first case, the segment bisector of $\overline{BE}$ coincides with the line $DS$, so $ABDE$ is a rhombus and $AB \parallel ED$.

In the second case, the segment bisector of $\overline{BE}$ intersects the line $AS$ at exactly one point. From the remarks we have given, that point must be $X$ and also $A$, so $A = X$. Then the triangle $BEA$ is equilateral, so $ABCD$ is a rhombus and $BC \parallel ED$.

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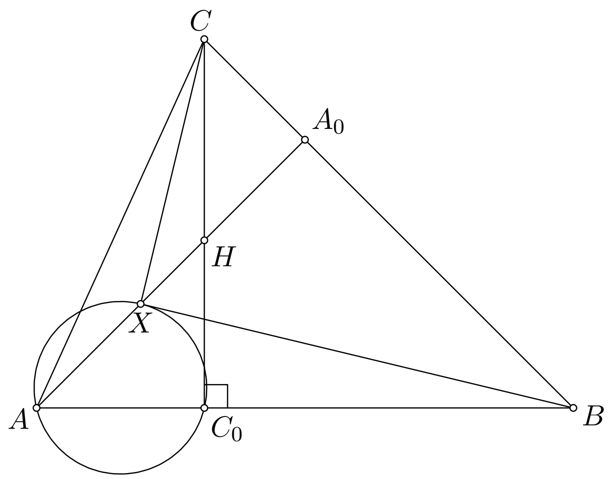
## T 6, (Michal Rolínek, Josef Tkadlec, Czech Republic)

Let $ABC$ be an acute triangle. Denote by $B_0$ and $C_0$ the feet of the altitudes from vertices $B$ and $C$, respectively. Let $X$ be a point inside the triangle $ABC$ such that the line $BX$ is tangent to the circumcircle of the triangle $AXC_0$ and the line $CX$ is tangent to the circumcircle of the triangle $AXB_0$. Show that the line $AX$ is perpendicular to $BC$.

### First solution




Let $A_0$ be the foot of the altitude from $A$. The quadrilateral $ACA_0C_0$ is cyclic because $\sphericalangle AA_0C = \sphericalangle AC_0C = 90°$. By the power of the point $B$ with respect to that circle we have $|BA||BC_0| = |BA_0||BC|$.

The power of the point $B$ with respect to the circumcircle of $AXC_0$ gives $|BX|^2 = |BA||BC_0|$.

Similarly, we have $|CX|^2 = |CA||CB_0| = |CA_0||BC|$.

Summing these two results we have

$$|BX|^2 + |CX|^2 = |BA_0||BC| + |CA_0||BC| = |BC|^2.$$

The converse of Pythagora's theorem implies $\sphericalangle BXC = 90°$.

Moreover, from $|BX|^2 = |BA_0||BC|$, i.e. $|BX| : |BC| = |BA_0| : |BX|$ we have that the triangles $BXA_0$ and $BCX$ are similar. It follows that

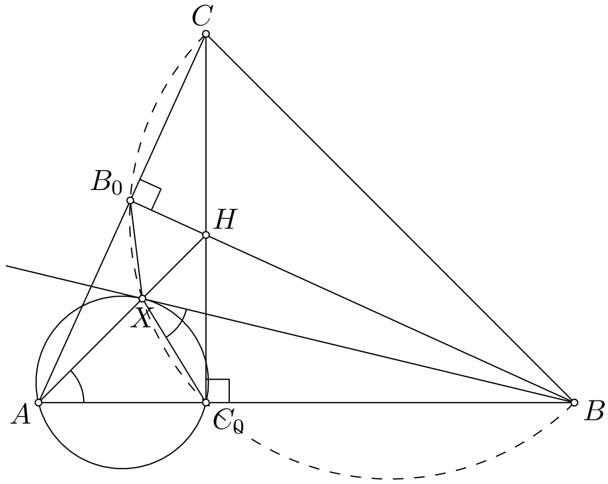
$$\sphericalangle BA_0X = \sphericalangle BXC = 90° = \sphericalangle BA_0A,$$

so $A_0, X$ and $A$ are collinear, so $AX$ and $BC$ are perpendicular.

## Second solution (by Tomislav Pejković, coordinator)

Let $H$ be the orthocenter of the triangle $ABC$. Because $BX$ is tangent to the circumcircle of $AXC_0$ we have $\sphericalangle BXC_0 = \sphericalangle BAX$ (the tangent chord angle theorem). Hence the triangles $BAX$ and $BXC_0$ are similar.

Analogously, the triangle $CAX$ and $CXB_0$ are similar.




Observe that the quadrilateral $AC_0HB_0$ is cyclic because $\sphericalangle AC_0H = \sphericalangle AB_0H = 90°$. The power of the point $B$ with respect to circumcircles of $AC_0X$ and $AC_0HB_0$ gives

$$|BB_0||BH| = |BA||BC_0| = |BX|^2.$$

From this we conclude that the triangles $BXH$ and $BB_0X$ are similar and $\sphericalangle BXH = \sphericalangle XB_0H = \sphericalangle XB_0C - 90°$. Since $CAX$ and $CXB_0$ are similar we have $\sphericalangle XB_0C = \sphericalangle AXC$. We obtained $\sphericalangle BXH = \sphericalangle AXC - 90°$ and analogously $\sphericalangle CXH = \sphericalangle AXB - 90°$.

Summing up these results we get

$$\sphericalangle BXC = \sphericalangle BXH + \sphericalangle CXH = \sphericalangle AXC + \sphericalangle AXB - 180° = 180° - \sphericalangle BXC$$

and so $\sphericalangle BXC = 90°$.

Hence, the points $B, C_0, X, B_0, C$ all lie on the same circle and we have

$$\sphericalangle AXB = \sphericalangle BC_0X = 180° - \sphericalangle XB_0B = 180° - \sphericalangle BXH$$

which means that $A, X$ and $H$ are collinear. So $AX$ and $BC$ are perpendicular.

**Third solution (by teams Croatia, Hungary and Poland)**

By power of the point we have

$$|CX|^2 = |CA||CB_0|, \quad |BX|^2 = |BA||BC_0|,$$

so the point $X$ is the intersection of the circle with center $C$ and radius $\sqrt{|CA||CB_0|}$ and the circle with center $B$ and radius $\sqrt{|BA||BC_0|}$. There are two such points, but only one is in the interior of the triangle $ABC$, so we conclude that the point $X$ is unique.

On the other hand we will prove that the point $Y$ which is the intersection of the circle with diameter $\overline{BC}$ and the altitude from the point $A$ has the same properties as the point $X$, from which we conclude that $X$ and $Y$ are the same point and hence $X$ lies on the line perpendicular to $BC$.

Since $\sphericalangle BB_0C = 90^\circ$, the quadrilateral $BCB_0Y$ is cyclic and hence $\sphericalangle CBB_0 = \sphericalangle CYB_0$. On the other hand $\sphericalangle CAY = 90^\circ - \sphericalangle ACB = \sphericalangle CBB_0 = \sphericalangle CYB_0$, so by the tangent-chord theorem the line $CY$ is tangent to the circumcircle of the triangle $AYB_0$. Analogously, the line $BY$ is tangent to the circumcircle of the triangle $AYC_0$. Hence, $X = Y$.

### I 4 (Kamil Duszenko, Poland)

Let $k$ and $m$, with $k > m$, be positive integers such that the number $km(k^2 - m^2)$ is divisible by $k^3 - m^3$. Prove that $(k - m)^3 > 3km$.

### First solution

Let $d$ be the greatest common divisor of $k$ and $m$. Write $k = da$, $m = db$. Then $a$ and $b$ are relatively prime. Moreover, $a > b$.

The number $km\left(k^2 - m^2\right) = d^4ab\left(a^2 - b^2\right) = d^4ab(a-b)(a+b)$ is divisible by $k^3 - m^3 = d^3(a^3 - b^3) = d^3(a-b)(a^2 + ab + b^2)$, so we have

$$a^2 + ab + b^2 \mid dab(a + b).$$

However, since the numbers $a$ and $b$ are relatively prime, the number $a^2+ab+b^2$ is relatively prime to $a$, $b$, and $a+b$. (For example, in case of $a+b$ we note that $a^2+ab+b^2 = (a+b)a+b^2$, and $a + b$ is relatively prime to $b$ and hence to $b^2$.) Thus

$$a^2 + ab + b^2 \mid d.$$

This, in particular, yields $d \geqslant a^2 + ab + b^2 = (a - b)^2 + 3ab > 3ab$. Therefore

$$(k - m)^3 = d^3(a - b)^3 \geqslant d^3 = d^2 \cdot d > d^2 \cdot 3ab = 3km.$$

### Second solution (by Wojciech Nadara, Poland)

Since $k^2 + km + m^2$ divides $km(k + m)$ and $(k^2 + km + m^2)(k + m)$ we have that it divides their difference $(k + m)(k^2 + m^2) = k^3 + k^2m + km^2 + m^3$. From this we conclude that $k^2 + km + m^2$ also divides $k^3 + k^2m + km^2 + m^3 - k(k^2 + km + m^2) = m^3$.

Analogously, we conclude that $k^2 + km + m^2$ divides $k^3$.

Multiplying the second power of $k^2 + km + m^2 \mid k^3$ with $k^2 + km + m^2 \mid m^3$ we conclude $(k^2 + km + m^2)^3 \mid k^6m^3$. Hence $k^2 + km + m^2$ also divides $k^2m$ and analogously $km^2$.

Adding all the results we have obtained we conclude that $k^2 + km + m^2$ divides $k^3 - 3k^2m + 3km^2 - m^3 = (k - m)^3$.

Because $k > m$, i.e. $k - m > 0$, we have $k^2 + km + m^2 \leqslant (k - m)^3$.

Since $(k - m)^2$ is equivalent to $k^2 + km + m^2 > 3km$, we obtain $3km < (k - m)^3$.

**T 7 (Mariusz Skaba, Poland)**

Let $A$ and $B$ be disjoint nonempty sets with $A \cup B = \{1, 2, 3, \ldots, 10\}$. Show that there exist elements $a \in A$ and $b \in B$ such that the number $a^3 + ab^2 + b^3$ is divisible by 11.

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**Solution**

For each $n = 0, 1, 2, \ldots$ the numbers $2^n, 2^{n+1}, 2^{n+2}, \ldots, 2^{n+9}$ have different remainders when divided by 11.

Suppose that for every $b \in B$ there is no $a \in A$ such that $a \equiv 2b \pmod{11}$.

From the above statement there exists $n \in \{0, 1, \ldots, 9\}$ such that $b \equiv 2^n \pmod{11}$ and we conclude that elements of $B$ give ten different remainders when divided by 11, so $B$ has 10 elements. That is a contradiction with the fact that $A$ is nonempty.

Therefore there exist $b \in B$ and $a \in A$ such that $a \equiv 2b \pmod{11}$, and we have

$$a^3 + ab^2 + b^3 \equiv 8b^3 + 2b^3 + b^3 = 11b^3 \equiv 0 \pmod{11}.$$

### T 8 (Aivaras Novikas, Lithuania)

We call a positive integer $n$ *amazing* if there exist positive integers $a, b, c$ such that the equality

$$n = (b, c)(a, bc) + (c, a)(b, ca) + (a, b)(c, ab)$$

holds. Prove that there exist 2011 consecutive positive integers which are amazing.
(By $(m, n)$ we denote the greatest common divisor of positive integers $m$ and $n$.)

### Solution

We may choose such positive integers $x_1, x_2, \dots, x_{2011}$ that the numbers

$$y_1 = x_1^2(x_1 + 2), \quad y_2 = x_2^2(x_2 + 2), \quad \dots, \quad y_{2011} = x_{2011}^2(x_{2011} + 2)$$

are pairwise coprime. For example, we may choose $x_1 = 1$ and $x_i = y_1 y_2 \dots y_{i-1} - 1$ for every consecutive $i$. This choice guarantees that for every integer $2 \leqslant i \leqslant 2011$ both $x_i$ and $x_i + 2$ (hence, $y_i$ as well) are coprime with any of the numbers $y_1, y_2, \dots, y_{i-1}$.

If a positive integer $n$ is divisible by any of the numbers $y_1, y_2, \dots, y_{2011}$ then it is amazing. Indeed, if, say, $n = y_i m = x_i^2(x_i + 2)m$ for some positive integers $m$ and $1 \leqslant i \leqslant 2011$ then $n = (b, c)(a, bc) + (c, a)(b, ca) + (a, b)(c, ab)$ for $a = m x_i^2$, $b = m x_i$, $c = x_i$.

Since the numbers $y_1, y_2, \dots, y_{2011}$ are pairwise coprime, the Chinese remainder theorem implies that there exists a positive integer $k$ satisfying the equalities

$$k \equiv -i \pmod{y_i}, \quad i = 1, 2, \dots, 2011.$$

This means that $k + i$ is divisible by $y_i$ for any $1 \leqslant i \leqslant 2011$. Thus, the consecutive positive integers $k + 1, k + 2, \dots, k + 2011$ are all amazing, and the statement of the problem is proved.