FIRST MIDDLE EUROPEAN MATHEMATICAL OLYMPIAD
EISENSTADT
AUSTRIA 2007

1. MEMO, Eisenstadt, Austria

Individual competition, September 22, 2007

1. Let a, b, c, d be positive real numbers with $a + b + c + d = 4$. Prove that

$$a^2bc + b^2cd + c^2da + d^2ab \leq 4.$$

2. A set of balls contains n balls which are labeled with numbers 1, 2, 3, …, n. Suppose we are given $k > 1$ such sets. We want to colour the balls with two colors, black and white, in such a way that

   (a) the balls labeled with the same number are of the same colour,

   (b) any subset of $k + 1$ balls with (not necessarily all different) labels a_1, a_2, …, a_{k+1} satisfying the condition $a_1 + a_2 + \ldots + a_k = a_{k+1}$, contains at least one ball of each colour.

   Find, depending on k, the greatest possible number n which admits such a colouring.

3. Let k be a circle and k_1, k_2, k_3 and k_4 four smaller circles with their centres O_1, O_2, O_3 and O_4 respectively on k. For $i = 1, 2, 3, 4$ and $k_5 = k_1$ the circles k_i and k_{i+1} meet at A_i and B_i such that A_i lies on k. The points O_1, A_1, O_2, A_2, O_3, A_3, O_4, A_4, lie in that order on k and are pairwise different. Prove that $B_1B_2B_3B_4$ is a rectangle.

4. Determine all pairs (x, y) of positive integers satisfying the equation

$$x! + y! = x^y.$$

Each problem is worth 8 points.

The order of the problems does not depend on their difficulty.

Time: 5 hours

Time for questions: 45 min