2nd MEMO, Olomouc, Czech Republic

Team competition, September 7, 2008

T–1: Find all functions $f : \mathbb{R} \to \mathbb{R}$ (so, f is a function from the real numbers to the real numbers) such that

$$xf(x + xy) = xf(x) + f(x^2)f(y)$$

for all real numbers x and y.

T–2: In a given n-tuple of positive integers with $n \geq 2$, we choose in each step a pair of numbers and replace each of them by their sum, i.e. we make the transformation

$$(\ldots, a, \ldots, b, \ldots) \to (\ldots, a + b, \ldots, a + b, \ldots).$$

Determine all values of n for which, after a finite number of steps, we can get an n-tuple of identical numbers from any initial n-tuple.

T–3: Given an acute-angled triangle ABC, let E be a point situated on the different side of the line AC than B, and let D be an interior point of the line segment AE. Suppose that $\angle ADB = \angle CDE$, $\angle BAD = \angle ECD$ and $\angle ACB = \angle EBA$. Prove that B, C and E are collinear.

T–4: Let n be a positive integer. Prove that if the sum of all positive divisors of n is a perfect power of 2, then the number of these divisors is also a perfect power of 2.

Each problem is worth 8 points.

The order of the problems does not depend on their difficulty.

Time: 5 hours

Time for questions: 45 min