language: English

3rd Middle European Mathematical Olympiad

Individual Competition
26th September, 2009

ProblemI-1.
Find all functions $f\colon \mathbb{R} \to \mathbb{R}$ such that

$$f(xf(y)) + f(f(x) + f(y)) = yf(x) + f(x + f(y))$$

for all $x, y \in \mathbb{R}$, where $\mathbb{R}$ denotes the set of real numbers.

ProblemI-2.
Suppose that we have $n \geqslant 3$ distinct colours. Let $f(n)$ be the greatest integer with the property that every side and every diagonal of a convex polygon with $f(n)$ vertices can be coloured with one of n colours in the following way:

- at least two distinct colours are used, and
- any three vertices of the polygon determine either three segments of the same colour or of three different colours.

Show that $f(n) \leqslant (n-1)^2$ with equality for infinitely many values of n.

ProblemI-3.
Let $ABCD$ be a convex quadrilateral such that AB and CD are not parallel and $AB = CD$. The midpoints of the diagonals AC and BD are E and F. The line EF meets segments AB and CD at G and H, respectively. Show that $\sphericalangle AGH = \sphericalangle DHG$.

ProblemI-4.
Determine all integers $k \geqslant 2$ such that for all pairs (m, n) of different positive integers not greater than k, the number $n^{n-1} - m^{m-1}$ is <u>not</u> divisible by k.

Time: 5 hours
Time for questions: 30 min
Each problem is worth 8 points.
The order of the problems does not depend on their difficulty.