language: English

3^{rd} Middle European Mathematical Olympiad

TEAM COMPETITION
27^{th} SEPTEMBER, 2009

Problem T-1.
Let x, y, z be real numbers satisfying $x^2 + y^2 + z^2 + 9 = 4(x + y + z)$. Prove that

$$x^4 + y^4 + z^4 + 16(x^2 + y^2 + z^2) \geqslant 8(x^3 + y^3 + z^3) + 27$$

and determine when equality holds.

Problem T-2.
Let a, b, c be real numbers such that for every two of the equations

$$x^2 + ax + b = 0, \quad x^2 + bx + c = 0, \quad x^2 + cx + a = 0$$

there is exactly one real number satisfying both of them.
Determine all the possible values of $a^2 + b^2 + c^2$.

Problem T-3.
The numbers $0, 1, 2, \ldots, n$ $(n \geqslant 2)$ are written on a blackboard. In each step we erase an integer which is the arithmetic mean of two different numbers which are still left on the blackboard. We make such steps until no further integer can be erased. Let $g(n)$ be the smallest possible number of integers left on the blackboard at the end. Find $g(n)$ for every n.

Problem T-4.
We colour every square of the 2009×2009 board with one of n colours (we do not have to use every colour). A colour is called *connected* if either there is only one square of that colour or any two squares of the colour can be reached from one another by a sequence of moves of a chess queen without intermediate stops at squares having another colour (a chess queen moves horizontally, vertically or diagonally). Find the maximum n, such that for every colouring of the board at least one colour present at the board is connected.

Problem T-5.
Let $ABCD$ be a parallelogram with $\sphericalangle BAD = 60^\circ$ and denote by E the intersection of its diagonals. The circumcircle of the triangle ACD meets the line BA at $K \neq A$, the line BD at $P \neq D$ and the line BC at $L \neq C$. The line EP intersects the circumcircle of the triangle CEL at points E and M. Prove that the triangles KLM and CAP are congruent.

Problem T-6.
Suppose that $ABCD$ is a cyclic quadrilateral and $CD = DA$. Points E and F belong to the segments AB and BC respectively, and $\sphericalangle ADC = 2\sphericalangle EDF$. Segments DK and DM are height and median of the triangle DEF, respectively. L is the point symmetric to K with respect to M. Prove that the lines DM and BL are parallel.

Problem T-7.
Find all pairs (m, n) of integers which satisfy the equation

$$(m+n)^4 = m^2n^2 + m^2 + n^2 + 6mn.$$

Problem T-8.
Find all non-negative integer solutions of the equation

$$2^x + 2009 = 3^y 5^z.$$

Time: 5 hours
Time for questions: 45 min
Each problem is worth 8 points.
The order of the problems does not depend on their difficulty.